Lecture 22:

M/G/∞ Queue, Superposition

Part I. M/G/∞ Queue.

Example 22.1. (M/G/∞ Queue)

Consider all the students talking on their cellphones. For the same reasoning as arrivals at the Tim Hortons at DC, the "beginnings of calls" can be viewed as "arrivals" which follow a Poisson Process with rate λ. For each student, assume the duration of their call are i.i.d. X with CDF of its distribution being G, i.e., $\mathbb{P}(X \le r) = G(r)$; and with mean being μ, i.e., $\mathbb{E}X = \mu$. Suppose at time 0, the number of calling is 0.

Q: What is the number of calls still in progress at time t?

A: The probability a call started at time s and ended by time t is

$$\mathbb{P}(X \leq t-s) = G(t-s).$$

So the probability a call started at s but still in progress at time t is

$$1 - G(t-s).$$

Thus the number of calls still in progress at time t is Poisson with mean

$$\lambda \int_{s=0}^{t} [1 - G(t-s)]\, ds = \lambda \int_0^t [1 - G(r)]\, dr.$$

change of variables
$r = t - s$

Theorem 22.1. Suppose the number of arrivals follows Poisson Process with rate λ, duration of each arrivals is i.i.d. X with CDF G and mean μ, then the number of service in progress at time t is Poisson$\left(\lambda \int_0^t [1-G(r)]\, dr\right)$.

Corollary 22.1. The number of service in progress in a long run is Poisson $(\lambda\mu)$.

Proof. Let $t = \infty$,

$$\lambda \int_0^\infty [1 - G(r)]\,dr = \lambda \int_0^\infty \mathbb{P}(X > r)\,dr$$
$$= \lambda \int_0^\infty \mathbb{E}[\mathbb{1}_{\{X>r\}}]\,dr$$
$$= \lambda\, \mathbb{E} \int_0^\infty \mathbb{1}_{\{r<X\}}\,dr$$
$$= \lambda\, \mathbb{E} \int_0^X 1\,dr$$
$$= \lambda\, \mathbb{E} X$$
$$= \lambda \mu. \quad \square$$

Remark 22.1. We assumed the system starts empty, since the number of initial calls still in the system at time t decreases to 0 as $t \longrightarrow \infty$, the limiting result is the same even if the system starts with any initials number of calls X_0.

Example 22.2. Customers arrival at a Sport Chek at rate 10 per hour, 60% of the customers are men and 40% are women. Suppose men stay in the store for an amount of time that is exponential with mean $\frac{1}{2}$ hour, while women for an amount of time that is uniformly distributed on $[0, \frac{1}{2}]$ hour.

Q: What is the probability in equilibrium that there are four men and two women in the store?

A: By Poisson thinning, the arrivals of men and women are independent Poisson Process with rate 6 and 4. Since the mean time in the store is $\frac{1}{2}$ for men and $\frac{1}{4}$ for women, by the Corollary 22.1, the number of men and women in equilibrium are independent Poisson with mean $6 \cdot \frac{1}{2} = 3$ and $4 \cdot \frac{1}{4} = 1$. Thus, the probability of interest is

$$e^{-3} \cdot \frac{3^4}{4!} \cdot e^{-1} \cdot \frac{1^2}{2!} = \frac{27}{16} e^{-4} .$$

Example 22.3. People arrive at a puzzle exhibit according to a Poisson Process with rate 2/min. The exhibit has enough copies of the puzzle. Suppose the puzzle takes an amount of time to solve that is uniform on $(0, 10)$ minutes.

Q(a): What is the distribution of people working on the puzzle in equilibrium?

A: Poisson (10).

Q(b): What is the probability that there are three people working on puzzles, one has been working more than 4 minutes, and two less than 4 minutes?

A: The probability a customer who arrived x minutes ago is still working on the puzzle is $\frac{10-x}{10}$.

Thus, the number that has been working more than 4 minutes is Poisson with mean

$$2 \cdot \int_4^{10} \frac{10-x}{10} \, dx = 3.6.$$

So the probability of interest is

$$e^{-3.6} \cdot \frac{3.6^1}{1!} \cdot e^{-6.4} \cdot \frac{6.4^2}{2!} = e^{-10} \cdot 73.728.$$

Part II. Superposition

Theorem 22.2. Suppose $\{N_1(t): t \geq 0\}$, $\{N_2(t): t \geq 0\}$, $\cdots$ $\{N_k(t): t \geq 0\}$ are independent Poisson Processes with rates $\lambda_1, \lambda_2, \ldots, \lambda_k$, then

$$\{N_1(t) + N_2(t) + \cdots + N_k(t): t \geq 0\}$$

is a Poisson Process with rate $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.

Example 22.4. Given a Poisson process of red arrivals with rate λ and an independent Poisson process of green arrivals with rate μ, what is the probability that we get 6 red arrivals before a total of 4 green ones?

A: First notice that this event is equivalent to having at least 6 red arrivals in the

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first 9. The reason is as follows.

①. If we have 6 or more red arrivals in the first 9, then we have 6 red arrivals before the forth green.

②. If not, then there are at most 5 red arrivals, in the first 9, then we have at least 4 green arrivals in the first 9. Thus the fourth green arrival is before the sixth red arrival.

Then by Theorem 22.2, one can view the red and green Poisson Processes as being constructed by starting with one Poisson Process with rate $\lambda+\mu$. Because red and green arrive independently, the colour of the combined arrivals can be decided by flipping coins with probability $p=\frac{\lambda}{\lambda+\mu}$ for the red.

So the probability of interest is

$$\sum_{k=6}^{9} \binom{9}{k} \cdot p^k \cdot (1-p)^{9-k} = \sum_{k=6}^{9} \binom{9}{k} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^k \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{9-k}$$

$$= \sum_{k=6}^{9} \frac{9!\, \lambda^k \cdot \mu^{9-k}}{k!(9-k)!(\lambda+\mu)^9} .$$

Method 2: The probability of interest is

$$\sum_{k=6}^{9} \frac{\mathbb{P}(\text{Poisson}(\lambda)=k) \cdot \mathbb{P}(\text{Poisson}(\mu)=9-k)}{\mathbb{P}(\text{Poisson}(\lambda+\mu)=9)}$$

$$= \sum_{k=6}^{9} \frac{e^{-\lambda} \cdot \left(\lambda^k/k!\right) \cdot e^{-\mu} \cdot \mu^{9-k}/(9-k)!}{e^{-(\lambda+\mu)} \cdot (\lambda+\mu)^9/9!}$$

$$= \sum_{k=6}^{9} \frac{9!\, \lambda^k \cdot \mu^{9-k}}{k!(9-k)!(\lambda+\mu)^9} .$$

This is the end of this lecture!

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